NOTES ON THE LOGICAL STRUCTURE OF THE THEORY OF RELATIVITY

1999-11-20 (from HPS 143) T.V. Daly S.J.

1. Einstein's Theory of Relativity stresses the fact that **movement** is relative, so that no absolute can be found on the level of motion, but it then leads us to discover **laws of physics** that are absolute in the truer sense of being verified, and so leads to a **deeper contact with reality**, since these laws cover facts not previously known.

To see how it does this we have to start by considering the advantage of finding **invariant expressions** for propositions we know to be true.

> Relativity physics is especially interested in invariants, and it has discovered and named a few more [besides "action" and "entropy"]. It is a common mistake to suppose that Einstein's theory of relativity asserts that everything is relative. Actually it says, "there are absolute things in the world but you **must look deeply for them**. The things **that first present themselves** to your notice are for the most part relative".
> (Eddington, *The nature of the Physical World*, 1928, p.23, emphasis added)

2. The same (true) **proposition** can have different **expressions**, when formulated in different **contexts**. For example: "I am here", "You are there", "He was there", can all refer to the same fact, though their context (of speaker, and of time of speaking) may vary.

3. You can **communicate** better if you are aware of the contexts and can make the required transformations. To do this you must understand

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the proposition. Many transformations will be based on common sense as in the above example, but more difficult cases can be helped by **rules**—like the lists of equivalents given in a French-English dictionary, or the more complex rules given in a French grammar.

Note that such transformation is often **required** if we are to **preserve** the truth. Thus, when I turn around, "the window is on my right" has to be changed to "the window is on my left".

4. When such transformations are made some expressions are found to remain **invariant.** These have special scientific value, as applying in a wider range of contexts. Thus, "Tom was behind the desk at the beginning of the class" covers each of the three contexts used in (2) above.

Exploring the context enables us to construct expressions with fuller invariance—so that they can be grasped readily by a wider variety of people. (See Frege, "The Thought", translated in *Mind*, 1956, p.289-311, especially on "I", p.297-99.)

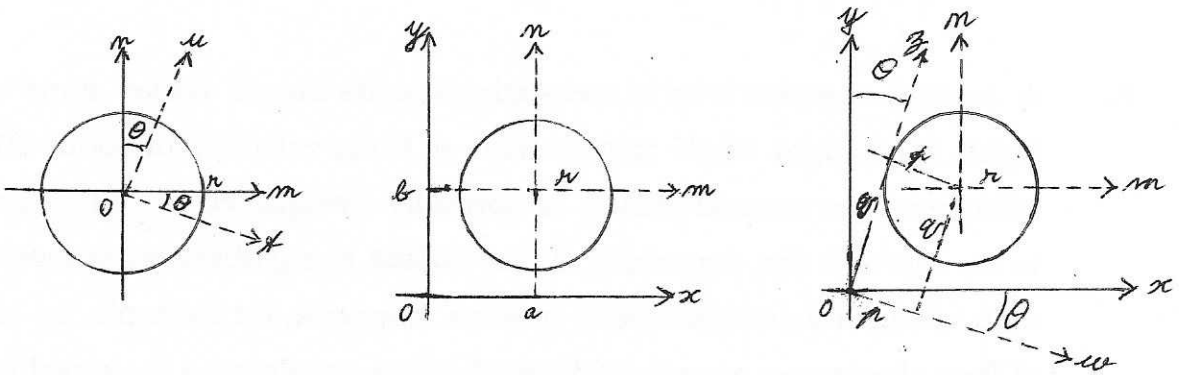
5. The example in (4) was of a **concrete fact**, but the same principle applies to **abstract laws**. For example, people in general would agree to "Water is liquid". To allow for Eskimos we transform this to "Water is liquid above 0°C." Allowing for boilermen: "Water is liquid between 0°C and 100°C". Allowing for mountain dwellers: "Water is liquid between 0°C and 100°C at a pressure of 760mm of mercury". From this we see the need for a formula **relating** the temperature and the pressure. They are not independent. Seeing the need does not, of course, give us the formula, but it is a big step towards finding it.

6. Some examples of transformations:

(a) A person going down in a lift with velocity "v" will have to transform vertical measurements "s" made by a stationary observer of events in the lift, according to the rule $d = s - vt$, to get his own downward measurements "d".

The law of fall $s = ut + (gt^2)/2$ becomes for him $d = (u - v)t + (gt^2)/2$, which preserves the same form. So this form of the law gives no information as to whether the observer is in the lift or not.

(b) In coordinate geometry, different "observers" can choose different reference frames. For two frames, m0n and t0u, with the same origin 0 but at an angle to each other, the equation of a circle with its centre at 0 will have the same mathematical form $m^2 + n^2 = r^2$, which is therefore invariant for this rotation of axes.



But for an observer with a frame (such as x0y) whose origin is not at the centre of the circle, the circle will be governed by a rule or equation of a different form, namely

$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2$. This can, however, be written as $(x - a)^2 + (y - b^2 = r^2$, and this equation applies for **any translation that** keeps the axes parallel, giving a form which enables

us to recognize a circle as distinct from an ellipse, a parabola, or a hyperbola.

This new, more complex, form of the equation can be shown to remain **invariant also for a rotation of axes**, as in the third diagram, where the frame w0z has been added.
We get the equation $(w - p)^2 + (z - q)^2 = r^2$. The relations between the constants are $p = a \cos \theta - b \sin \theta$

$q = a \sin \theta + b \cos \theta$.

The **law of transformation** for this rotation of the axes is

$w = x \cos \theta - y \sin \theta$, or, in reverse, $x = z \sin \theta + w \cos \theta$

$z = x \sin \theta + y \cos \theta$ $\qquad$ $y = z \cos \theta - w \sin \theta$

Note that, in this law of transformation, the value of z depends, not only on y, but on both x and y.

7. A human observer who is measuring speeds has to make some use of **light**. But light itself moves with a finite velocity, of about 300,000 kilometres per second (which is normally designated by "c"). This has to be allowed for, especially if the **object** being observed is moving with a velocity comparable to his—as happens, for example, in the case of **free electrons,** especially when they are accelerated in a cyclotron. A transformation that allows for this must be applied to the laws of physics (and of geometry) that we use for moving objects.

8. One possible **transformation that allows for this** is the Lorentz transformation (also discovered previously by Voight). It is **not imposed** on us, since any attempt to prove it directly would require some measurement of the speed of light, and so would involve arguing in a circle. It can only be commended above alternative hypotheses by **its results**.

9. This indirect verification involves the **principle of equivalence**, that "physical principles and laws are the same for all observers" (see Lonergan, *Insight*, p.65 [1st edition p.40]. The value of the principle is indicated in points (1) and (2) above.

10. The **Lorentz transformation** from a reference frame using the measures x for distance and t for time, to another frame using the measures x' and t' and moving in the direction of x with velocity v follows the rules:

$$x' = H(x - vt) \text{ and } t' = H(t - vx/c^2),$$

where $H = 1/\sqrt{(1 - v^2/c^2)}$.

11. Under this hypothesis the **speed of light** is invariant, that is, its measure c, is the same for all observers, even if one is moving faster than another. This fits in with **Michelson and Morley's** failure to detect any difference in the speed of light even when the earth was moving in different directions.

There is also a **geometrical invariant** in the measure of

small distance: $ds^2 = dx^2 - c^2dt^2$

(or, more generally, $ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2$).

12. Note that **if both** distance and time vary the transformation must take account of both of them **together**, just as w depends on both x and y in point (6) above, and liquidity depends on both pressure and temperature in point (3).

This **association** of measures of distance and time led to the introduction of the term "space-time" by Minkowski. It is basically no more mysterious than the association of assorted measures (like temperature, mass, charge, pressure) in **other equations** in physics.

It should cause philosophical difficulties only for an empiricist who feels that space and time are the most basic realities, rather than just **aspects** of a more complex real world through which that world makes contact with our senses.

13. When we apply the Lorentz transformation to the **laws of physics**, such as Newton's law that the mass by the acceleration is equal to the applied force, $ma = F$ (or $d(mv)/dt = F$, that is, the force equals the rate of change of momentum), and the expression $mv^2/2$ for kinetic energy, we find that the expressions of these laws are **not** invariant. This suggests that though they give us the truth for activities in which the finite velocity of light is not important they do **not** express a **full understanding** of what is going on.

14. So it is worth **looking for fuller expressions** of physical laws. You cannot always derive these. You just have to **invent** them, trying out many suggestions (which is, of course, the way that most important scientific laws were found, such as Kepler's laws and Newton's inverse square law of gravity).

15. But we have a heuristic device. We examine only those laws that prove to be **invariant**. Since not many algebraical equations will in fact remain invariant under the Lorentz transformation, the possible objects of our choice are drastically limited. These laws must also give the same results as the well-tested newtonic laws for bodies that are moving with a velocity that is small compared to the velocity of light.

Note that these will be **new laws**, not just transformations of laws that were previously known.

16. One such relationship that does remain invariant is $F/H = d(m\,v)/dt$, where H is the same as in (10) above).

17. This **replaces** Newton's law, since it reduces to it in the special case where v/c is very small. In that case H = 1, so the expression becomes $F = d(mv)/dt$. But for things moving with a very high velocity this new law gives results which are very different from those given by Newton's laws.

18. Another invariant expression is mHc^2. For small velocities this is equal to $mc^2 + mv^2/2$.

Since c is a constant, mc^2 is a constant, and thus invariant, so we can define **a new concept** by means of the expression $mHc^2 - mc^2$. For small velocities this reduces to $mv^2/2$, which is the ordinary Newtonian kinetic energy, so a reasonable name for our new concept is "relativistic kinetic energy".

We can give similar names to the other new concepts:

Relativistic Force	F/H
Relativistic Momentum	mHv
Relativistic Mass	mH
Relativistic Energy	$mHc^2 - mc^2$

19. Using these new concepts we get a set of laws that are **analogous** to the Newtonian laws, that are **invariant** under the Lorentz transformation, and that reduce to the Newtonian laws on the assumption of an infinite velocity for light.

Since they reduce to the Newtonian laws, they are **verified** by all the evidence that goes to verify the Newtonian laws.

Hence they do **not** show that those laws were false.

They are, rather, a **development** from the Newtonian laws—extending them so as to be able to cover much wider fields of data. Note that this is different from being a **deduction** from Newton's laws—this they certainly are not, as should be clear from the account given above, especially in points (13) to (15).

20. These laws will enable us to make **new predictions** about the motion of particles that have a very high velocity. Such predictions have been thoroughly **verified**, especially in the case of **electrons** accelerated by a cyclotron. Such electrons have been given energy of 300 million electron volts; this would require a speed of 35 times the velocity of light under the old laws, but a speed less than the velocity of light (by about 1 kilometre per second) under the new ones.

21. Moreover, being more complex, these new laws **suggest further connections** in nature that would not have been thought of before. For instance, from the way it was introduced in number (17) above, it is tempting to think of the expression mc^2 as giving the "rest energy" of a particle. There is a possibility of further empirical verification here.

22. Note that these laws can be used **for a single observer**. They are **not** in any way limited to cases of two different **observers** moving with different velocities, even though we had to consider the possibility of such observers when we were starting out to look for invariants.

Nor do these laws **deal with** transformations; they deal with ordinary physical processes. They **were found** by considering the exigencies of transformations. That is, they were found by seeking laws that could have the same expression for as many different observers as possible.

23. The experimental verification of the relativistic laws, mentioned in (19) to (21) above, is not exactly a verification of the theory of relativity. But this verification of the laws shows that the relativity considerations were **needed**. And it shows that the **principle of equivalence** is useful.

Note that our discarding of an absolute in space (and in motion) brings us closer to an absolute in **understanding**.

The new laws give us **deeper understanding** of the physical world.

They bring us closer to **reality**.

Thus we have been brought closer to reality, even though we started off by rejecting the idea that reality could be found in **mere absolute space**.

24. What do we mean by "**reality**" ? It is something important, and it is the opposite to **illusion**. It is almost identical with **truth,** which is the opposite to **error**, and with **fact**, which is the opposite to **fiction**.

Most people of practical common sense, as well as most scientists, and even most lawyers, would seem to agree—once it is pointed out—that illusions (like errors and fictions) tend to be dispelled by persistent inquiry. And, on the other hand, reality (like truth and fact) is what stands up to intelligent questioning.

25. The desire to find an absolute in **mere** space or time by an isolated look or touch is the fruit of a lazy empiricism, and of a neglect of what "reality" really means. Einstein's **relativity** is almost the opposite to **relativism**. It leads us to absolutes in the form of verified scientific laws.

26. Since the only case of the principle of equivalence that we have discussed in detail here is that in which two observers are moving with constant velocity relative to one another, and since the Lorentz transformation was introduced to deal with this case, the rest of our discussion, after point (10) has been concerned with the **Special Theory** of Relativity.

A wider use of the principle of equivalence would allow extensions to be made that cover other cases. This is what Einstein did in the **General Theory of Relativity**. Lonergan introduces this in *Insight* Ch.5, section 2. The "tensor calculus" which he mentions there (147/170-171) is an extremely powerful heuristic device going beyond that mentioned in point 15 above.